Useful Fourier Transform results in the study of Visual Motion

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1 Introduction

This tutorial note includes some results from Fourier theory which are useful in the study of Visual Motion. In particular, we derive the spectral properties of a image subject uniform translational motion, which is a classic result used to develop insight into the more general case.

Additionally some results regarding multidimensional directional derivatives are also presented. These often crop up in the context of models of spatio-temporal filters used in early vision. These results are useful for understanding various classes of rotation invariant filters which have gained some popularity in the motion estimation community.

Finally, the Fourier transform of a Gaussian function is derived in full detail. This is an annoyingly simple result which nevertheless seldom finds its way into textbooks.

2 Uniform Translational Motion

Let $I_0(x, y)$ denote an intensity image with continuous spatial variables x, y. Assume that the intensity image undergoes uniform translational motion with horizontal velocity component V_x and vertical velocity component V_y. The resulting spatiotemporal intensity function $I(x, y, t)$ is related to the still frame $I_0(x, y)$ according to,

$$I(x, y, t) = I_0(x - V_x t - x_0, y - V_y t - y_0), \tag{1}$$

where x_0, y_0 denote the horizontal and vertical displacement of $I(x, y)$ at $t = 0$. For convenience it is assumed that $x_0 = y_0 = 0$ so,

$$I(x, y, t) = I_0(x - V_x t, y - V_y t). \tag{2}$$

3 The Fourier Domain Representation of Uniform Translational Motion

The simple case of uniform translational motion admits a closed-form spatiotemporal frequency domain description which provides useful insight into the motion problem. Let ω_x, ω_y and ω_t denote the two spatial frequency and one temporal frequency variable respectively. We seek an expression for $\hat{I}(\omega_x, \omega_y, \omega_t) = \mathcal{F}\{I(x, y, t)\}$, where $\mathcal{F}\{\cdot\}$ denotes the Fourier transform operation.

From Equation (2) we have $I(x,y,t) = I_0(x - V_x t, y - V_y t)$ so,

$$
\begin{aligned}
\hat{I}(\omega_x, \omega_y, \omega_t) &= \mathcal{F}\{I(x,y,t)\} \\
&= \iiint_{-\infty}^{+\infty} I(x,y,t)\; e^{-i(\omega_x x + \omega_y y + \omega_t t)}\; \mathrm{d}x\; \mathrm{d}y\; \mathrm{d}t \\
&= \iiint_{-\infty}^{+\infty} I_0(x - V_x t, y - V_y t)\; e^{-i(\omega_x x + \omega_y y + \omega_t t)}\; \mathrm{d}x\; \mathrm{d}y\; \mathrm{d}t \\
&\quad \text{Letting } u = x - V_x t \text{ and } v = y - V_y t, \\
&= \iiint_{-\infty}^{+\infty} I_0(u,v)\; e^{-i(\omega_x (u + V_x t) + \omega_y (v + V_y t) + \omega_t t)}\; \mathrm{d}u\; \mathrm{d}v\; \mathrm{d}t \\
&= \iiint_{-\infty}^{+\infty} I_0(u,v)\; e^{-i(\omega_x u + \omega_x V_x t + \omega_y v + \omega_y V_y t + \omega_t t)}\; \mathrm{d}u\; \mathrm{d}v\; \mathrm{d}t \\
&= \int_{-\infty}^{+\infty} \left\{ \iint_{-\infty}^{+\infty} I_0(u,v)\; e^{-i(\omega_x u + \omega_y v)}\; \mathrm{d}u\; \mathrm{d}v \right\} e^{-i(\omega_x V_x t + \omega_y V_y t + \omega_t t)}\; \mathrm{d}t \\
&= \int_{-\infty}^{+\infty} \hat{I}_0(\omega_x, \omega_y) e^{-i(\omega_x V_x t + \omega_y V_y t + \omega_t t)}\; \mathrm{d}t \\
&= \hat{I}_0(\omega_x, \omega_y) \int_{-\infty}^{+\infty} e^{-i(\omega_x V_x + \omega_y V_y) t} e^{-i\omega_t t}\; \mathrm{d}t \\
&= \hat{I}_0(\omega_x, \omega_y)\, \delta(\omega_x V_x + \omega_y V_y + \omega_t). \qquad (3)
\end{aligned}
$$

Equation (3) is a product of two terms:

- $\hat{I}_0(\omega_x, \omega_y)$

  This is the Fourier transform of the still image $I_0(x,y)$. In the spatiotemporal frequency space $\omega_x, \omega_y, \omega_t$, $\hat{I}(\omega_x, \omega_y)$ is constant for all ω_t.

- $\delta(\omega_x V_x + \omega_y V_y + \omega_t)$

  This term describes a plane in spatiotemporal frequency space. The orientation of the plane in spatiotemporal frequency space is determined by the horizontal and vertical motion velocities. The normal vector of the plane is given by $[V_x, V_y, 1]^\top$.

Thus the spectral energy of the spatiotemporal intensity $I(x,y,t)$ is constrained to lie in a plane in spatiotemporal frequency space. In particular, the product in Equation (3) constrains the spectral energy to the plane

$$[\omega_x, \omega_y, \omega_t][V_x, V_y, 1]^\top = \omega_x V_x + \omega_y V_y + \omega_t = 0,$$

with the value at a point $\omega_x, \omega_y, \omega_t$ in the plane given by the value of $\hat{I}_0(\omega_x, \omega_y)$.

3.1 Temporal Aliasing

One interesting consequence of the Fourier domain expression for uniform translational motion is that an image, spatially sampled so as to satisfy Nyquist's criterion can result in frequency domain aliasing when the signal undergoes translational motion.

4 Selected Fourier Transform Properties

4.1 Differentiation Properties

Let $f(x)$ and $f(\omega)$ be a Fourier transform pair, then we may express $f(x)$ in terms of $F(\omega)$ using the inverse Fourier transform relation,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{iwx}\, \mathrm{d}\omega.$$

Differentiating both sides yields,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{iwx}\,\mathrm{d}\omega\right)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}i\omega F(\omega)e^{iwx}\,\mathrm{d}\omega,\end{aligned}$$

from which we note that,

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x) \Longleftrightarrow i\omega F(\omega). \tag{4}$$

Similarly, starting with the Fourier transform formula,

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\mathrm{d}x$$

and differentiating with respect to ω yields the result,

$$\frac{\mathrm{d}}{\mathrm{d}\omega}F(\omega) \Longleftrightarrow -ixf(x) \tag{5}$$

or equivalently, $i\frac{\mathrm{d}}{\mathrm{d}\omega}F(\omega) \Longleftrightarrow xf(x)$.

In general, with $\boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^N$, if $f(\boldsymbol{x}) \Longleftrightarrow F(\boldsymbol{\omega})$ represents a Fourier transform pair then, for $k \in \{1, 2, \ldots, N\}$ and $p_k \in \{0, 1, \ldots, N\}$ such that

$$1 \leq \sum p_k \triangleq \sum_{k=1}^{N} p_k \leq N$$

then,

$$\frac{\partial^{\Sigma p_k}}{\prod_{k=1}^{N}\partial x_k^{p_k}} f(\boldsymbol{x}) \Longleftrightarrow \prod_{k=1}^{N}(i\omega_k)^{p_k}F(\boldsymbol{\omega}) \tag{6}$$

and

$$\frac{\partial^{\Sigma p_k}}{\prod_{k=1}^{N}\partial \omega_k^{p_k}} F(\boldsymbol{\omega}) \Longleftrightarrow \prod_{k=1}^{N}(-ix_k)^{p_k}f(\boldsymbol{x}). \tag{7}$$

4.2 Directional Derivatives

It is also useful to determine Fourier transform relations for directional derivatives. Let $f(\boldsymbol{x}) \Longleftrightarrow F(\boldsymbol{\omega})$ represent a Fourier transform pair with $\boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^N$. Let $\hat{\boldsymbol{u}}$ be a unit vector in $\mathbb{R}^N$ and let $\mathcal{D}_{\hat{\boldsymbol{u}}}$ denote the derivative operator in the direction $\hat{\boldsymbol{u}}$. With $\mathcal{F}\{\cdot\}$ representing the Fourier transform we have,

$$\begin{aligned}\mathcal{F}\{\mathcal{D}_{\hat{\boldsymbol{u}}}f(\boldsymbol{x})\} &= \mathcal{F}\{[\hat{\boldsymbol{u}}\cdot\nabla]f(\boldsymbol{x})\}\\ &= \mathcal{F}\left\{\sum_{k=1}^{N}u_k\frac{\partial}{\partial x_k}f(\boldsymbol{x})\right\}\\ &= \sum_{k=1}^{N}u_k\ \mathcal{F}\left\{\frac{\partial}{\partial x_k}f(\boldsymbol{x})\right\}\\ &= \sum_{k=1}^{N}u_k\left\{iw_kF(\boldsymbol{\omega})\right\}\\ &= i[\hat{\boldsymbol{u}}\cdot\boldsymbol{\omega}]F(\boldsymbol{\omega}).\end{aligned}$$

We thus we conclude that,

$$\mathcal{D}_{\hat{\boldsymbol{u}}}f(\boldsymbol{x}) \Longleftrightarrow i[\hat{\boldsymbol{u}}\cdot\boldsymbol{\omega}]F(\boldsymbol{\omega}). \tag{8}$$

We arrive at a similar relation for the directional derivative in the Fourier domain as,

$$\mathcal{D}_{\hat{\boldsymbol{u}}} F(\boldsymbol{\omega}) \Longleftrightarrow -i[\hat{\boldsymbol{u}} \cdot \boldsymbol{x}] f(\boldsymbol{x}). \tag{9}$$

For higher order directional derivatives, we note that $\mathcal{D}^n_{\hat{\boldsymbol{u}}} = [\hat{\boldsymbol{u}} \cdot \nabla]^n$ so that,

$$\begin{aligned} \mathcal{F}\left\{\mathcal{D}^n_{\hat{\boldsymbol{u}}} f(\boldsymbol{x})\right\} &= \mathcal{F}\left\{[\hat{\boldsymbol{u}} \cdot \nabla]^n f(\boldsymbol{x})\right\} \\ &= \mathcal{F}\left\{\left[\sum_{k=1}^{N} u_k \frac{\partial}{\partial x_k}\right]^n f(\boldsymbol{x})\right\}. \end{aligned}$$

Here the expression in the brackets is an n^{th} order polynomial containing all combinations of mixed partial derivatives. Using Equation (6) the Fourier transform of the product of each of these terms with $f(\boldsymbol{x})$ may be expressed in the frequency domain and the resulting terms collected to yield the result,

$$\mathcal{D}^n_{\hat{\boldsymbol{u}}} f(\boldsymbol{x}) \Longleftrightarrow i^n [\hat{\boldsymbol{u}} \cdot \boldsymbol{\omega}]^n F(\boldsymbol{\omega}). \tag{10}$$

Similarly,

$$\mathcal{D}_{\hat{\boldsymbol{u}}} F(\boldsymbol{\omega}) \Longleftrightarrow (-i)^n [\hat{\boldsymbol{u}} \cdot \boldsymbol{x}]^n f(\boldsymbol{x}). \tag{11}$$

4.3 The Fourier Transform of a Gaussian

In this section we show that the Fourier transform of a Gaussian is also Gaussian. First we will require the result that the Gaussian density integrates to unity,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \mathrm{d}x = 1. \tag{12}$$

This result may be shown as follows,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \mathrm{d}x &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \mathrm{d}x \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} \mathrm{d}y \ \text{ with } \ y = 0 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \mathrm{d}x \ \mathrm{d}y \\ &= \int_{0}^{2\pi} \int_{0}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \ r \ \mathrm{d}r \ \mathrm{d}\theta \\ &= \int_{0}^{2\pi} -\frac{1}{2\pi} \left[\int_{0}^{\infty} e^{-\frac{r^2}{2\sigma^2}} \cdot -\frac{r}{\sigma^2} \mathrm{d}r\right] \mathrm{d}\theta \\ &= \int_{0}^{2\pi} -\frac{1}{2\pi} \left[e^{-\frac{r^2}{2\sigma^2}}\right]_0^{\infty} \mathrm{d}\theta \\ &= \int_{0}^{2\pi} \frac{1}{2\pi} \ \mathrm{d}\theta \\ &= 1. \end{aligned}$$

We will use this result to show that the Fourier transform of a Gaussian in also a Gaussian (up to a scale factor) but the variance σ^2 of which is equal to the *reciprocal* of the variance of the original Gaussian. Specifically, the Fourier transform relation we wish to show is given by,

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \Longleftrightarrow e^{-\frac{\omega^2}{2/\sigma^2}}. \tag{13}$$

Let $F(\omega)$ denote the Fourier transform of the function $f(x)$. With $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$ we have,

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} e^{-i\omega x} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2 - i\omega x} \mathrm{d}x \end{aligned}$$

We may apply the technique of completing the square whereby we rewrite a quadratic equation $ax^2 + bx + c$ in the alternate form,

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \tag{14}$$

which, after some algebraic manipulation, yields

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x + i\omega\sigma^2\right)^2 - \frac{1}{2}\omega^2\sigma^2} \mathrm{d}x \\ &= e^{-\frac{1}{2}\omega^2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left(x + i\omega\sigma^2\right)^2} \mathrm{d}x \\ &= e^{-\frac{\omega^2}{2/\sigma^2}}. \end{aligned}$$

The integral in the second to last line is equal to unity by our earlier result.